Formulae for half a side!

In a spherical triangle ABC, Prove that

(i) $\sin\frac{a}{2} = \sqrt{\frac{-\cos S\ \cos(S-A)}{\sin B.\ \sin C}}$

(ii) $\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B).\ \cos(S-C)}{\sin B.\ \sin C}}$

(iii) $\tan\frac{a}{2} = \sqrt{\frac{-\cos S.\ \cos(S-A)}{\cos(S-B).\ \cos(S-C)}}$

(iv) $\sin a = \frac{2N}{\sin B.\ \sin C}$

Where $N^2 = -\cos S.\ \cos(S-A).\ \cos(S-B).\ \cos(S-C)$

& $2S = A+B+C$.

Proof of (i) We have $2\sin^2\frac{a}{2} = 1-\cos a$

$$= 1 - \frac{\cos A + \cos B.\cos C}{\sin B.\sin C}$$

$$= \frac{\sin B.\sin C - \cos A - \cos B.\cos C}{\sin B.\sin C}$$

$$= \frac{-\cos A - (\cos B.\cos C - \sin B.\sin C)}{\sin B.\sin C}$$

$$= -\frac{\cos A + \cos(B+C)}{\sin B.\sin C}$$

$$= -\frac{2\cos\left(\frac{B+C+A}{2}\right).\cos\left(\frac{B+C-A}{2}\right)}{\sin B.\sin C}$$

$$= -\frac{2\cos\left(\frac{2S}{2}\right).\ \cos\left(\frac{A+B+C-2A}{2}\right)}{\sin B.\sin C}$$

$$\Rightarrow \cancel{2}\sin^2\frac{a}{2} = -\frac{\cancel{2}\cos S.\ \cos(S-A)}{\sin B.\sin C}$$

$$\therefore \sin\frac{a}{2} = \sqrt{\frac{-\cos S.\ \cos(S-A)}{\sin B.\ \sin C}}$$

$$= \frac{-\cos(B+C) - \cos A}{\cos A + \cos(B-C)} = - \frac{\cos A + \cos(B+C)}{\cos A + \cos(B-C)}$$

$$= - \frac{2\cos\frac{A+B+C}{2} \cdot \cos\frac{A-B-C}{2}}{2\cos\frac{A+B-C}{2} \cdot \cos\frac{A-B+C}{2}}$$

$$= - \frac{2\cos\frac{2S}{2} \cdot \cos -\left(\frac{B+C-A}{2}\right)}{2\cos\left(\frac{A+B+C-2C}{2}\right) \cdot \cos\left(\frac{A+B+C-2B}{2}\right)}$$

$$= - \frac{\cancel{2}\cos S \cdot \cos\left(\frac{B+C+A-2A}{2}\right)}{\cancel{2}\cos\left(\frac{2S-2C}{2}\right) \cdot \cos\left(\frac{2S-2B}{2}\right)}$$

$$= - \frac{\cos S \cdot \cos(S-A)}{\cos(S-C) \cdot \cos(S-B)}$$

$$\therefore \tan\frac{a}{2} = \sqrt{\frac{-\cos S \cdot \cos(S-A)}{\cos(S-B) \cdot \cos(S-C)}}$$

Proof of (iv) We have $\sin a = 2\sin\frac{a}{2} \cdot \cos\frac{a}{2} = 2 \cdot \sqrt{\frac{-\cos S \cdot \cos(S-A)}{\sin B \cdot \sin C}} \cdot \sqrt{\frac{\cos(S-B) \cdot \cos(S-C)}{\sin B \cdot \sin C}}$

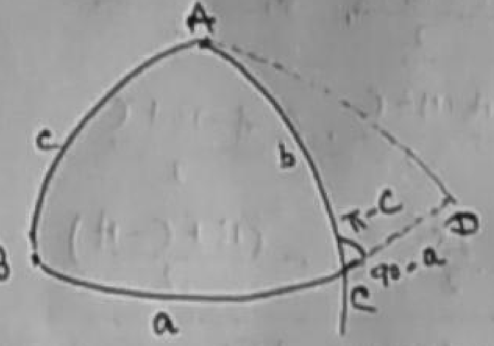
$$= \frac{2}{\sin B \cdot \sin C} \cdot \sqrt{-\cos S \cdot \cos(S-A) \cdot \cos(S-B) \cdot \cos(S-C)}$$

$$= \frac{2N}{\sin B \cdot \sin C}.$$

Where $N^2 = -\cos S \cdot \cos(S-A) \cdot \cos(S-B) \cdot \cos(S-C)$.

Sine- Cosine Formula:

"Relation between three sides and two angles of a spherical triangle."



Let ABC be any spherical triangle. Produce the side BC of the triangle to D so that $BD = 90^\circ$

Then $CD = 90^\circ - a$ & $\angle ACD = 180^\circ - C$

Join A and D by great circle arc.

Applying Cosine formula in spherical triangle ADB.

$$\cos AD = \cos c \cdot \cos 90^\circ + \sin c \cdot \sin 90^\circ \cdot \cos B$$

$$= \cos c \cdot 0 + \sin c \cdot 1 \cdot \cos B$$

$$\Rightarrow \cos AD = \sin c \cdot \cos B \quad \text{———} \quad (1)$$

Again applying Cosine formula in spherical triangle ADC

$$\cos AD = \cos b \cdot \cos(90^\circ - a) + \sin b \cdot \sin(90^\circ - a) \cdot \cos(\pi - C)$$

$$= \cos b \cdot \sin a + \sin b \cdot \cos a \cdot (-\cos C)$$

$$\Rightarrow AD = \cos b \cdot \sin a - \sin b \cdot \cos a \cdot \cos C \quad \text{———} \quad (2)$$

From (1) & (2), we have

$$\sin c \cdot \cos B = \cos b \cdot \sin a - \sin b \cdot \cos a \cdot \cos C$$

$$\Rightarrow \boxed{\sin c \cdot \cos B = \sin a \cdot \cos b - \cos a \cdot \sin b \cdot \cos C}$$

It is the required formula.